

Solving Virasoro Constraints in Matrix Models

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Abstract

This is a brief review of recent progress in constructing solutions to the matrix model Virasoro equations. These equations are parameterized by a degree n polynomial $W_n(x)$, and the general solution is labeled by an arbitrary function of $n - 1$ coefficients of the polynomial. We also discuss in this general framework a special class of (multi-cut) solutions recently studied in the context of $\mathcal{N} = 1$ supersymmetric gauge theories.

Introduction. It was realized in the beginning of nineties that matrix models partition functions typically satisfy an infinite set of Virasoro-like equations [1, 2]. These were nothing but Ward identities (Schwinger-Dyson equations) which mainly fixed matrix model partition functions (because of the topological nature of matrix models [3], the Ward identities were restrictive enough). Moreover, it turned out that one of the most technically effective ways to deal with matrix models was to solve these Virasoro equations (they are also sometimes called loop equations) [4, 5, 6, 7, 8].

At early times of matrix models one usually dealt with Virasoro equations describing relatively simple “phases” so that the equations had unambiguous solutions. An interest to more complicated phases of matrix models has revived after G.Bonnet, F.David and B.Eynard [9] proposed to deal with the multi-cut (or multi-support) solutions (known for long, [10, 11, 6, 12]) in a new way: releasing the tunneling constraint [11]. Their approach was later applied by R.Dijkgraaf and C.Vafa [13] to description of low energy superpotentials in $\mathcal{N} = 1$ SUSY gauge theories, [14].

More concretely, the authors of [14] considered the $\mathcal{N} = 2$ SUSY gauge (Seiberg-Witten) theory in special points where some BPS states become massless. Therefore, these states can condense in the vacuum which breaks half of the supersymmetries (leading to $\mathcal{N} = 1$ SUSY) and gives rise to a non-zero superpotential. Values of this superpotential in minima are related to the prepotential of a Seiberg-Witten-like theory. In turn, R.Dijkgraaf and C.Vafa associated [13] the prepotential with logarithm of a partition function of the Hermitean one-matrix model in the planar limit of multi-cut solutions (it was later proved in [15]).

In fact, actual definition of the multi-cut partition functions is a separate problem. For instance, one could simply define them as (arbitrary) solutions to the corresponding Virasoro equations (D-module point of view). Then, one may ask what is special about the concrete Dijkgraaf-Vafa (DV)

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solutions. In the present paper we make further steps in this direction, discuss the space of all solutions to the Virasoro equations in the multi-cut phase and show that the DV solutions form a basis in this space. They are distinguished by a special property of isomonodromy that allows one to associate with these Seiberg-Witten-like systems a Whitham hierarchy [15, 16], the corresponding partition function having a multi-matrix model integral representation [9, 17].

Hermitean one-matrix model. Hermitean one matrix model is given by the formal matrix integral over $N \times N$ Hermitean matrix M

$$Z_W(t) \equiv \frac{1}{\text{Vol}_{U(N)}} \int DM \exp \frac{1}{g} \left(-\text{Tr } W(M) + \text{Tr} \sum_k t_k M^k \right) \quad (1)$$

Here $W(x)$ is an arbitrary function that we usually assume to be a polynomial of degree $n+1$, $W_n(x) \equiv \sum_k^{n+1} T_k x^k$ and the constant g^2 controls the genus expansion. This integral still needs to be defined. One possibility is to substitute it with its saddle point approximations [9, 17]. Different saddle-points $M = M_0$ are given by the equation $W'(M_0) = 0$. If the polynomial

$$W'(x) = \prod_{i=1}^n (x - \alpha_i) \quad (2)$$

has roots α_i , then, since M_0 are matrices defined modulo $U(N)$ -conjugations (which allow one to diagonalize any matrix and permute its eigenvalues), the different saddle points are represented by

$$M_0 = \text{diag}(\alpha_1, \dots, \alpha_1; \alpha_2, \dots, \alpha_2; \dots; \alpha_n, \dots, \alpha_n) \quad (3)$$

with α_i appearing N_i times, $\sum_{i=1}^n N_i = N$. In fact, there is no need to keep these N_i non-negative integers: in final expressions (like formulas for the multidensities and prepotentials) they can be substituted by any complex numbers. Moreover, N_i can depend on T_k (i.e. on the shape of $W(\phi)$) and g .

Now, using at intermediate stage the eigenvalue representation of matrix integrals, one can rewrite [9, 17] the matrix integral (1) over $N \times N$ matrix M as n -matrix integral over $N_i \times N_i$ matrices M_i (each obtained with the shift by α_i : just changing variables in the matrix integral (1)), which is nothing but the DV solution [13]

$$\begin{aligned} Z_W^{(matr)}(t|M_0) &\sim \int \prod_{i=1}^n DM_i \exp \left(\sum_{i,k} \text{Tr } t_k^{(i)} M_i^k \right) \prod_{i < j}^n \alpha_{ij}^{2N_i N_j} \times \\ &\times \exp \left(2 \sum_{k,l=0}^{\infty} (-)^k \frac{(k+l-1)!}{\alpha_{ij}^{k+l} k! l!} \text{Tr}_i M_i^k \text{Tr}_j M_j^l \right) \end{aligned} \quad (4)$$

The variables $t_k^{(i)}$ are given by the relation

$$\sum_{k=0}^{\infty} t_k \left(\sum_{i=1}^n \text{Tr}_i (\alpha_i + M_i)^k \right) = \sum_{i=1}^n \left(\sum_{k=0}^{\infty} t_k^{(i)} \text{Tr}_i M_i^k \right) \quad (5)$$

with arbitrary $N_i \times N_i$ matrices M_i .

Virasoro constraints. The other possibility is to observe that (1) satisfies the infinite set of (Virasoro) equations (=Schwinger-Dyson equations,=Ward identities) [2]

$$\begin{aligned} \hat{L}_m Z_W(t) &= 0, \quad m \geq -1 \\ \hat{L}_m &= \sum_{k \geq 0} k (t_k - T_k) \frac{\partial}{\partial t_{k+m}} + g^2 \sum_{\substack{a+b=m \\ a,b \geq 0}} \frac{\partial^2}{\partial t_a \partial t_b} \end{aligned} \quad (6)$$

$$\frac{\partial Z_W}{\partial T_k} + \frac{\partial Z_W}{\partial t_k} = 0 \quad \forall k = 0, \dots, n+1 \quad (7)$$

and call any solution to these equations the matrix model partition function. Then, the partition function is not a function but a formal D -module, i.e. the entire collection of power series (in t -variables), satisfying a system of consistent linear equations. Solution to the equations does not need to be unique, however, an appropriate analytical continuation in t -variables transforms one solution to another, and, on a large enough moduli space (of *coupling constants* t), the whole entity can be considered, at least, formally as a single object: this is what we call *the partition function*. Naively different solutions are interpreted as different *branches* of the partition function, associated with different *phases* of the theory. Further, solutions to the linear differential equations can be often represented as integrals (over *spectral varieties*), but integration “contours” remain unspecified: they can be generic *chains* with complex coefficients (in the case of integer coefficients this is often described in terms of *monodromies*, but in the case of partition functions the coefficients are not restricted to be integer). A *model* of partition function is an integral formula which has enough many free parameters to represent the generic solution of the differential equations in question.

A familiar example that could clarify these notions is provided with the cylindric functions. Their defining equation is

$$\left[\lambda^2 \partial_\lambda^2 + \lambda \partial_\lambda + (\lambda^2 - \nu^2) \right] Z_\nu(\lambda) = 0 \quad (8)$$

and an integral representation is

$$Z_\nu(x) = \frac{1}{2\pi} \int_C e^{-ix \sin \theta + i\nu \theta} d\theta \quad (9)$$

The model is given by the generic linear combination of two contours, say, chosen as in 8.423 of [18] (this choice fixes as the basis the Hankel functions).

Loop equations. Another form of the Virasoro equations (the loop equation) is produced by rewriting the infinite set of these equations through the unique generating function of all single trace correlators

$$\rho^{(1)}(z|t) \equiv \hat{\nabla}(z)\mathcal{F}, \quad \hat{\nabla}(z) \equiv \sum_{k \geq 0} \frac{1}{z^{k+1}} \frac{\partial}{\partial t_k}, \quad \mathcal{F} \equiv g^2 \log Z_W \quad (10)$$

Introducing notations $v(z)$ for $\sum_k t_k z^k$ and $[\dots]_+$ ($[\dots]_-$) for the projector onto non-negative (negative) degrees of z , one obtains *the loop equation* [4]

$$W'(z)\rho^{(1)}(z|t) = \left(\rho^{(1)}(z|t) \right)^2 + f_W(z|t) + g^2 \hat{\nabla}(z)\rho^{(1)}(z|t) + \left[v'(z)\rho^{(1)}(z|t) \right]_- \quad (11)$$

$$f_W(z|t) \equiv \left[W'(z)\rho^{(1)}(z|t) \right]_+ \equiv \hat{R}(z)\mathcal{F} \quad (12)$$

In order to consider (connected) multi-trace correlators, one needs to introduce higher generating functions (also named loop mean, resolvent etc) which we call multi-density

$$\rho^{(m)}(z_1, \dots, z_m|t) \equiv \hat{\nabla}(z_1) \dots \hat{\nabla}(z_m)\mathcal{F} \quad (13)$$

In what follows, we consider solutions to the Virasoro equations (6) as a formal series in t -variables, as well as a series in the coefficient g^2 (genus expansion):

$$\log Z_W = g^{-2} \mathcal{F} = \sum_{p \geq 0} g^{2p-2} \mathcal{F}^{(p)} \quad (14)$$

Main results.

Here we are going to review briefly the main results of the papers [7] and [8], where we defined the matrix model partition function as any solution to the Virasoro equations (6). In forthcoming paragraphs we briefly comment on these results.

- Any solution to the Virasoro constraints (taken as a formal series in t -variables and in genus expansion) is unambiguously labeled by an arbitrary function of n of $n + 2$ T -variables: the *bare* all genera prepotential.
- There is an **evolution operator** that generates from the t -independent bare prepotential the matrix model partition function which depends on t -variables and satisfies the Virasoro equations. This evolution operator *does not depend* on the choice of the arbitrary function, but only on T - and t -variables.
- One may invariantly define “the occupation numbers” of [9, 13] as eigenvalues of operators constructed from the evolution operator, formula (35) below. The corresponding DV solutions are described as eigenfunctions of these operators.
- (**Conjecture 1**) The evolution operator can be completely expressed in terms of the unique operator

$$\check{y} \equiv \sqrt{W'(x)^2 - 4g^2\check{R}(x)}, \quad \check{R}(x) \equiv - \sum_{a,b=0} (a+b+2)T_{a+b+2}x^a \frac{\partial}{\partial T_b} \quad (15)$$

its derivatives and $W'(x)$.

- (**Conjecture 2**) The evolution operator is constructed as a formal series in t with operator coefficients acting on the bare prepotential. These coefficients generate the full matrix model correlators. These operator coefficients are related to **operator multidensities** (13) exactly as the full correlators are related to the connected correlators, only an appropriate ordering prescription should be applied. This relation is **universal**, i.e. is the same for the Gaussian (quadratic) and non-Gaussian potentials.
- (**Conjecture 3**) The ordering used in the previous conjecture is not uniquely defined.

Solving the Virasoro constraints. In order to convert (11) into a solvable set of recurrent relations, we expand $\rho^{(1)}(z|t)$ in powers of g^2 and t 's

$$\begin{aligned} \rho^{(1)}(z|t) &= \sum_{p,m \geq 0} \frac{g^{2p}}{m!} \oint \dots \oint v(z_1) \dots v(z_m) \rho^{(p|m+1)}(z, z_1, \dots, z_m), \\ f_W(z|t) &= \sum_{p,m \geq 0} \frac{g^{2p}}{m!} \oint \dots \oint v(z_1) \dots v(z_m) f_W^{(p|m+1)}(z|z_1, \dots, z_m) \end{aligned} \quad (16)$$

In this way, we introduce the full set of multidensities $\rho_W^{(p|m)}(z_1, \dots, z_m)$ and auxiliary polynomials $f_W^{(p|m+1)}(z|z_1, \dots, z_m)$ (which distinguishes between different phases for a given $W(z)$) at zero t 's.

Acting on eq.(11) with the operator $\hat{\nabla}(z_1) \dots \hat{\nabla}(z_m)$ and putting all $t_k = 0$ afterwards, we obtain

a double-recurrent (in p and m) relation for the multidensities $\rho^{(p|m)}$

$$\begin{aligned} & W'(z)\rho^{(p|m+1)}(z, z_1, \dots, z_m) - f_W^{(p|m+1)}(z|z_1, \dots, z_m) = \\ &= \sum_q \sum_{m_1+m_2=m} \rho^{(q|m_1+1)}(z, z_{i_1}, \dots, z_{i_{m_1}})\rho^{(p-q|m_2+1)}(z, z_{j_1}, \dots, z_{j_{m_2}}) + \\ &+ \sum_{i=1}^m \frac{\partial}{\partial z_i} \frac{\rho^{(p|m)}(z, z_1, \dots, z_i, \dots, z_m) - \rho^{(p|m)}(z_1, \dots, z_m)}{z - z_i} + \hat{\nabla}(z)\rho^{(p-1|m+1)}(z, z_1, \dots, z_m) \end{aligned} \quad (17)$$

Together with (12) this relation is enough to find explicit expressions for all the multidensities through $W(z)$ and $f_W^{(p|1)}(z)$. In fact, the latter polynomials (all of degree $n-1$) are not independent, since for $m=0$,

$$f_W^{(p|1)}(z) = \check{R}F^{(p)}[T] \quad (18)$$

expresses all the f -polynomials through a single function of T (i.e. of $W(z)$) and g , the prepotential at $t=0$,

$$\mathcal{F}(t=0, g) = F[g, T] = \sum_{p=0}^{\infty} g^{2p} F^{(p)}[T] \equiv Z[g, T] \quad (19)$$

The operator \check{R} here is given in (15) and contains derivatives with respect to the T -variables. We call such operators *check operators* and denote by the “check” sign to distinguish them from *the hat operators*, which contain t -derivatives and are denoted by “hats”.

Note that the T_k dependence of $F[g, T]$ is not arbitrary, since the first two (\hat{L}_{-1} and \hat{L}_0) Virasoro constraints are linear in derivatives and can be consistently truncated to $t=0$ and then allow one to express two derivatives, say, $\partial F/\partial T_{n+1}$ and $\partial F/\partial T_n$ through $\partial F/\partial T_l$ with $l=0, \dots, n-1$. As a corollary, the partition function can be represented as

$$Z(T)|_{t=0} = Z[g, T] = \int dk z(k; \eta_2, \dots, \eta_n; \hbar) e^{\frac{1}{\hbar}(kx - k^2 w)} \quad (20)$$

with an arbitrary function z of n arguments $(k, \eta_2, \dots, \eta_n)$ and \hbar . Here the \hat{L}_{-1} -invariant variables are used,

$$\begin{aligned} w &= \frac{1}{n+1} \log T_{n+1}, \quad x = T_0 + \dots \sim \eta_{n+1}, \\ \eta_k &= \left(T_n^k + \frac{k(k-2)!}{n!} \sum_{l=1}^{k-1} (-)^l \frac{(n+1)^l (n-l)!}{(k-l-1)!} T_{n-l} T_n^{k-l-1} T_{n+1}^l \right) T_{n+1}^{-\frac{kn}{n+1}} \end{aligned} \quad (21)$$

As an immediate corollary of (17), we obtain for $p=0$ and $m=0$

$$\rho^{(0|1)}(z) = \frac{W'(z) - y(z)}{2} \quad (22)$$

with

$$y^2(z) = (W'(z))^2 - 4f_W^{(0|1)}(z) \quad (23)$$

Evolution check operator. The basic claim is that, for any $W(z)$, there is an evolution (check) operator $\check{U}_W(t)$, which converts *any* function $Z[T]$ of T_0, \dots, T_{n-1} (with prescribed dependence on T_n and T_{n+1}) into $Z_W(t) = \check{U}_W(t)Z[T]$, which satisfies the Virasoro constraints, $L_m(t)Z_W(t) = 0$, $m \geq -1$. This means that the evolution operator is the same for any values of the arbitrary parameters f (or for any function $Z[T]$) once $W(z)$ is fixed and that “orbits” of the evolution operators are completely parameterized by $W(z)$. Moreover, if $Z[T] = \sum_a c_a Z^{(a)}[T]$, then $Z_W(t) = \sum_a c_a Z_W^{(a)}(t)$. This means that one may arbitrarily choose a basis in the space of all solutions, with the evolution not disturbing the expansion of any solution w.r.t. this basis.

One may construct the operator $\check{U}_W(t)$ with the following procedure. For given T 's, we make use of the Virasoro constraints $\hat{L}_m Z(t) = 0$ and their multiple t -derivatives to recurrently express

$$\left. \frac{\partial^p Z}{\partial t_{k_1} \dots \partial t_{k_p}} \right|_{t=0} \text{ with all } 0 \leq k_i < \infty \text{ through } g^{2s} \left. \frac{\partial^{p+s} Z}{\partial T_{l_1} \dots \partial T_{l_{p+s}}} \right|_{t=0} \text{ with all } 0 \leq l_j < n$$

$$\left. \frac{\partial^p Z}{\partial t_{k_1} \dots \partial t_{k_p}} \right|_{t=0} = \sum_{\substack{s \\ l_1, \dots, l_{p+s}}} g^{2s} \mathcal{D}_{k_1 \dots k_p}^{l_1 \dots l_{p+s}} \left. \frac{\partial^{p+s} Z}{\partial T_{l_1} \dots \partial T_{l_{p+s}}} \right|_{t=0} \quad (24)$$

This is a straightforward procedure, and the sum over s is finite, from 0 to the integer part of $\left\{ \frac{k}{n-1} \right\}$: the expression for $\left. \frac{\partial Z}{\partial t_k} \right|_{t=0}$ contains $\left. \frac{\partial^2 Z}{\partial t_a \partial t_b} \right|_{t=0}$, but with $a, b \leq k - n - 1$, further, the expression for $\left. \frac{\partial^2 Z}{\partial t_a \partial t_b} \right|_{t=0}$ contains $\left. \frac{\partial^3 Z}{\partial t_a \partial t_b \partial t_c} \right|_{t=0}$, this time with $a, b, c \leq k - 2n - 2$ and so on.

Let us now define the operators

$$\check{D}_{k_1 \dots k_p}^{(p)} = \sum_{\substack{s \\ 0 \leq l_1, \dots, l_{p+s} \leq n-1}} g^{2s} \mathcal{D}_{k_1 \dots k_p}^{l_1 \dots l_{p+s}} \left. \frac{\partial^{p+s} Z}{\partial T_{l_1} \dots \partial T_{l_{p+s}}} \right|_{t=0} \quad (25)$$

and construct the evolution operator $\check{U}_W(t)$ as a series in these \check{D} -operators

$$\check{U}_W(t) = 1 + t_k \check{D}_k^{(1)} + \frac{1}{2} t_k t_l \check{D}_{kl}^{(2)} + \frac{1}{6} t_k t_l t_m \check{D}_{klm}^{(3)} + \dots \quad (26)$$

The fact that, for any $Z[T]$,

$$\hat{L}_m(t) Z_W(t) = \hat{L}_m(t) \check{U}_W(t) Z[T] = 0 \quad (27)$$

or, simply, that

$$\begin{aligned} \hat{L}_m(t) \check{U}_W(t) &= \left(\sum_k k T_k \check{D}_{k+m}^{(1)} + \sum_{a+b=m} \check{D}_{ab}^{(2)} \right) + \sum_l t_l \left(l \check{D}_{l+m}^{(1)} + \sum_k k T_k \check{D}_{k+m, l}^{(2)} + \sum_{a+b=m} \check{D}_{abl}^{(3)} \right) + \\ &+ \frac{1}{2} \sum_{l_1, l_2} t_{l_1} t_{l_2} \left(l_1 \check{D}_{l_1+m, l_2}^{(2)} + l_2 \check{D}_{l_2+m, l_1}^{(2)} + \sum_k k T_k \check{D}_{k+m, l_1, l_2}^{(3)} + \sum_{a+b=m} \check{D}_{abl_1 l_2}^{(4)} \right) + \dots = 0 \end{aligned} \quad (28)$$

is equivalent to vanishing of all the linear combinations of operators in brackets, and these are the characteristic equations for the \check{D} -operators.

Basis in the space of all solutions. One can now cleverly choose some basis in the space of all solutions. Note that the DV solutions do form such a basis. To have them written in a more clever way, one may present the contribution of a particular extremum M_0 (labeled by the set of N_i) above in the Givental-style decomposition form, expressing it through the product of n Gaussian partition functions ($Z_G^M(t|N)$ given by the $N \times N$ matrix integral with quadratic $W_G(x) \equiv \mathcal{M}x^2$), with its own N_i and $\mathcal{M}_i = W''(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j)$,

$$Z_W^{(matr)}(t|M_0) = \frac{\prod_{i=1}^n e^{-N_i W(\alpha_i)} \text{Vol}_{U(N_i)}}{\text{Vol}_{U(N)}} \prod_{i < j}^n \alpha_{ij}^{2N_i N_j} \prod_{i < j}^n \hat{\mathcal{O}}_{ij} \prod_{i=1}^n \hat{\mathcal{O}}_i \prod_{i=1}^n Z_G^{\mathcal{M}_i}(t^{(i)}|N_i) \quad (29)$$

with operators

$$\hat{\mathcal{O}}_{ij} = \exp \left(2 \sum_{k, l=0}^{\infty} (-)^k \frac{(k+l-1)!}{\alpha_{ij}^{k+l} k! l!} \frac{\partial}{\partial t_k^{(i)}} \frac{\partial}{\partial t_l^{(j)}} \right), \quad (30)$$

$\alpha_{ij} = \alpha_i - \alpha_j$, and

$$\hat{\mathcal{O}}_i = \exp \left(- \sum_{k \geq 3} \frac{W^{(k)}(\alpha_i)}{k!} \frac{\partial}{\partial t_k^{(i)}} \right) \quad (31)$$

$(W^{(k)}(x) = \partial_x^k W(x))$.

These DV solutions have a series of properties that basically have much to do with isomonodromic deformations (by $W(x)$) and are in charge of Whitham integrable systems behind the DV solutions [15, 16], see the end of the next paragraph.

Evolution operator as a function of \check{y} . So far we basically considered the “connected” correlation functions, $\rho^{(\cdot|m)}(z_1, \dots, z_m; g)$. The other possibility is to deal with “full” correlation functions,

$$K^{(\cdot|m)}(z_1, \dots, z_m; g) = Z_W(t; g)^{-1} \hat{\nabla}(z_1) \dots \hat{\nabla}(z_m) Z_W(t; g) \Big|_{t=0} = \sum_{p=0}^{\infty} g^{2(p-m)} K^{(p|m)}(z_1, \dots, z_m) \quad (32)$$

These are related by

$$\begin{aligned} K^{(\cdot|m)}(z_1, \dots, z_m; g) = \\ = \sum_{\sigma} \sum_{k=1}^m \sum_{\nu_1, \dots, \nu_k=1}^{\infty} \sum_{p_1, \dots, p_{\nu}=0}^{\infty} g^{2(p_1 + \dots + p_{\nu} - \nu)} \left(\sum_{\substack{m_1, \dots, m_k \\ m = \nu_1 m_1 + \dots + \nu_k m_k}} \frac{1}{\nu_1! (m_1!)^{\nu_1} \dots \nu_k! (m_k!)^{\nu_k}} \times \right. \\ \left. \times \rho^{(p_1|\tilde{m}_1)}(z_{\sigma(1)}, \dots, z_{\sigma(\tilde{m}_1)}) \rho^{(p_2|\tilde{m}_2)}(z_{\sigma(\tilde{m}_1+1)}, \dots, z_{\sigma(\tilde{m}_2)}) \dots \rho^{(p_{\nu}|\tilde{m}_{\nu})}(z_{\sigma(m-\tilde{m}_{\nu}+1)}, \dots, z_{\sigma(m)}) \right) \end{aligned} \quad (33)$$

Our task is to express the correlation functions defined in (32) and (13) with the help of hat-operators through the action of check-operators. In fact, this task is already solved for (32): one can easily see that

$$\begin{aligned} \check{K}^{(\cdot|m)}(z_1, \dots, z_m) = \sum_{k_1, \dots, k_m} \frac{1}{z_1^{k_1+1} \dots z_m^{k_m+1}} \check{D}_{k_1 \dots k_m}^{(m)}, \\ K^{(\cdot|m)}(z_1, \dots, z_m; g) = Z(T; g)^{-1} \check{K}^{(\cdot|m)}(z_1, \dots, z_m) Z(T; g) \end{aligned} \quad (34)$$

Moreover, manifest examples that can be found in [8] show that these quantities are expressed through \check{y} (15), its derivatives and $W'(x)$.

Note that one may now invariantly define the quantities that emerged in the DV solutions, $S_i \equiv \frac{N_i}{g}$. To this end, one should introduce the “occupation number” operators (in fact, these operators describe the monodromy of $\check{K}^{(\cdot|1)}(z)$)¹

$$g\check{S}_i \equiv \check{N}_i \equiv \oint_{\alpha_i} \check{K}^{(\cdot|1)}(z) dz \quad (35)$$

Then, S_i are nothing but the eigenvalues of this operator. This is analogous to the condition $\frac{\partial Z}{\partial T_0} = \frac{N}{g} Z$ which one usually adds to (6) in order to describe matrix integrals. Now one can define the DV solutions as eigenfunctions of the set of operators \check{S}_i . Therefore, S_i being eigenvalues, by definition, do not depend on T_k ’s. This exactly expresses the isomonodromic properties of the DV solutions, see, e.g., [6, 15, 12, 19]).

¹Relations like

$$\left[g^2 \oint_{A_i} \check{K}, g^2 \oint_{B_j} \check{K} \right] \stackrel{?}{\sim} g^2 \left[\check{S}_i, \frac{\partial}{\partial \check{S}_j} \right] = g^2 \left[\check{N}_i, \frac{\partial}{\partial \check{N}_j} \right] = g^2 \delta_{ij}$$

should serve as operator counterparts of the celebrated

$$\begin{aligned} S_i &= \oint_{A_i} \rho^{(0|1)} \\ \frac{\partial F_{DV}}{\partial S_i} &= \oint_{B_i} \rho^{(0|1)} \end{aligned}$$

for the particular DV solution. However, one should be careful about regularization, higher-loop corrections etc.

Check ρ -operators. The connected correlators ρ are more "fundamental" than the full K . Therefore, it is natural to wonder if one can find check-operator analogues of ρ 's, once we see that check-operator counterparts of K do exist and can be of some use. This means that we would like to put

$$\check{K}^{(\cdot|1)}(z; g) = \sum_{p=0}^{\infty} g^{2p-2} \check{\rho}^{(p|1)}(z; g) \quad (36)$$

In this way, one gets rid of the terms with explicitly present operators \check{R} and prepotentials $F^{(p)}$, the relevant check-operators $\check{\rho}^{(\cdot|p)}$ are expressed through $\check{y}(z; g)$ only (with the single exception of $\check{\rho}^{(0|1)}(z; g)$, which also contains $W'(z)$.) Thus, the check-operator $\check{K}^{(\cdot|p)}$ is a polynomial in W' of degree p . Instead, the g dependence is now distributed between explicit factors like g^{2p-2} and an additional g -dependence of $\check{y}(z; g)$. This, however, allows check-operators $\check{\rho}^{(p|m)}$ to look exactly the same (modulo ordering) as the corresponding Gaussian multidensities $\rho_G^{(p|m)}$, which are all expressed through y_G only.

In [8] we suggested a *hypothesis* that **eq.(36) is indeed true in all orders in g^2 and, moreover, similar expansions hold for all $\check{K}^{(\cdot|m)}(z_1, \dots, z_m; g)$: they can be all expressed through multilinear combinations of check operators $\check{\rho}^{(p|m)}$, which (for $(p|m) \neq (0|1)$) depend only on $\check{y}(z; g)$ and its z -derivatives in exactly the same way as $\rho_G^{(p|m)}$ depends on $y_G(z)$.** However, even to formulate this hypothesis, one needs to introduce some ordering prescription for products of check-operators which is not, as usual, unique. Different ordering prescriptions lead to different explicit formulas for $\check{\rho}$, and our hypothesis states that there exist orderings, when these expressions contain \check{y} , its derivative and nothing else, except for a few W' , see [8] for more possible choices.

In principle, when introducing $\check{\rho}$ -operators, we have different possibilities of definition, preserving one or another kind of their relation to \check{K} 's. They could be defined similarly to (34) from (13), so that eq.(38) below becomes an equality. However, it appears more interesting *instead* of preserving the equations, to require for $\check{\rho}^{(\cdot|k)}$ to be the same (up to ordering) as the Gaussian functions $\rho_G^{(\cdot|k)}$. We can construct recursively an operator modification of expression (33). Since the recurrent equations for \check{K} are linear

$$W'(z)K^{(\cdot|m+1)}(z, z_1, \dots, z_m) - \check{R}(z)K^{(\cdot|m)}(z_1, \dots, z_m) + \sum_{i=1}^m \frac{\partial}{\partial z_i} \frac{K^{(\cdot|m)}(z, z_1, \dots, z_i, \dots, z_m) - K^{(\cdot|m)}(z_1, \dots, z_m)}{z - z_i} = g^2 K^{(\cdot|m+2)}(z, z, z_1, \dots, z_m) \quad (37)$$

they coincide with the equations for operators \check{K} . The equations for functions ρ are not linear. Thus, for the operators $\check{\rho}$ we should choose the order in which different $\check{\rho}$ stand in the products.

Note that, with our definition,

$$\rho_W^{(p|m)}(z_1, \dots, z_m) \neq Z(T; g)^{-1} \check{\rho}_W^{(p|m)}(z_1, \dots, z_m; g) Z(T; g) \quad (38)$$

and, in variance with the l.h.s. of (38), its r.h.s. is still g -dependent.

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